# Pendulum dynamics with horizontal and vertical excitation

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The following describes the dynamics of a pendulum (a rigid body) with both horizontal and vertical accelerations applied at the support. This is a model for problems of much larger dimension I am interested in. We use the state vector

$$
\begin{aligned}
x_1 &= \Delta : \text{ vertical displacement} \\
x_2 &= \theta : \text{ rotation} \\
x_3 &= \dot{\Delta} \\
x_4 &= \dot{\theta}
\end{aligned}
$$

to describe the dynamics. The input accelerations in the horizontal and vertical directions are $u_x$ and $u_z$. The differential equation of motion is

$$
\begin{aligned}
m\ddot{\Delta} - mh\sin(\theta)\ddot{\theta} - mh\cos(\theta)\dot{\theta}^2 + k_{\mathrm{v}}\Delta + mg(1+u_z(t)) + c_{\mathrm{v}}\dot{\Delta} &= 0 \\
-mh\sin(\theta)\ddot{\Delta} + I\ddot{\theta} + k_{\mathrm{r}}\theta - mg(1+u_z(t))h\sin(\theta) + mgu_x(t)h\cos(\theta) + c_{\mathrm{r}}\dot{\theta} &= 0
\end{aligned}
$$

Abbreviate these two equations as

$$
F\left(t, \begin{pmatrix}\Delta \\ \theta\end{pmatrix}, \begin{pmatrix}\dot{\Delta} \\ \dot{\theta}\end{pmatrix}, \begin{pmatrix}\ddot{\Delta} \\ \ddot{\theta}\end{pmatrix}\right) = 0
$$

The residual for IDA is

$$
\mathrm{res}(t, x, \dot{x}) = \begin{pmatrix} \dot{x}(1:2) - x(3:4) \\ F\left(t, x(1:2), x(3:4), \dot{x}(3:4)\right) \end{pmatrix}
$$

The Jabobian of the residual contains the matrices

$$
M = \begin{bmatrix} \frac{\partial F_1}{\partial \ddot{\Delta}} & \frac{\partial F_1}{\partial \ddot{\theta}} \\ \frac{\partial F_2}{\partial \ddot{\Delta}} & \frac{\partial F_2}{\partial \ddot{\theta}} \end{bmatrix}; \quad C = \begin{bmatrix} \frac{\partial F_1}{\partial \dot{\Delta}} & \frac{\partial F_1}{\partial \dot{\theta}} \\ \frac{\partial F_2}{\partial \dot{\Delta}} & \frac{\partial F_2}{\partial \dot{\theta}} \end{bmatrix}; \quad K = \begin{bmatrix} \frac{\partial F_1}{\partial \Delta} & \frac{\partial F_1}{\partial \theta} \\ \frac{\partial F_2}{\partial \Delta} & \frac{\partial F_2}{\partial \theta} \end{bmatrix}
$$

Solving a linear equation with the Jacobian reduces to solving a linear equation with the matrix

$$
c_j^2 M + c_j C + K
$$

which has a sparse structure corresponding to a finite element stiffness matrix, hence my interest in using a custom linear solver with IDA.